Homework 2
MTH 829 Complex Analysis

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Proposition 0.1 (Exercise III.5.2)**.** *Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. If ϕ is a linear-fractional transformation, then*

$$(z_1, z_2; z_3, z_4) = (\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4))$$

Proof. There exists a unique fractional linear transformation ψ such that

$$\psi(z_1) = \infty \qquad \psi(z_2) = 0 \qquad \psi(z_3) = 1$$

and then by definition $(z_1, z_2; z_3, z_4) = \psi(z_4)$. Similarly, there exists a unqiue fractional linear transformation η such that

$$\eta \circ \phi(z_1) = \infty \qquad \eta \circ \phi(z_2) = 0 \qquad \eta \circ \phi(z_3) = 1$$

and by definition $(\phi(z_1), \phi(z_2); \phi(z_3), \phi(z_4)) = \eta \circ \phi(z_4)$. The composition $\eta \circ \phi$ is a fractional linear transformation that agrees with ψ on three points, so by uniqueness, $\psi = \eta \circ \phi$. Thus $\psi(z_4) = \eta \circ \phi(z_4)$, so the cross ratios are equal. □

Definition 0.2. *Two linear-fractional transformations ϕ_1, ϕ_2 are **conjugate** if there is a linear-fractional transformation ψ such that $\phi_2 = \psi\phi_1\psi^{-1}$.*

Proposition 0.3 (Exercise III.6.2)**.** *All translations, except the identity transformation, are mutually conjugate.*

Proof. Let ϕ_1, ϕ_2 be the translations $\phi_1(z) = z + \beta_1$ and $\phi_2(z) = z + \beta_2$ with $\beta_1, \beta_2 \in \mathbb{C} \setminus \{0\}$. Let ψ be the linear fractional transformation

$$\psi(z) = \frac{\beta_2}{\beta_1} z$$

Then

$$\psi\phi_1\psi^{-1}(z) = \psi\phi_1 \left(\frac{\beta_1}{\beta_2} z \right) = \psi \left(\frac{\beta_1}{\beta_2} z + \beta_1 \right) = \frac{\beta_2}{\beta_1} \left(\frac{\beta_1}{\beta_2} z + \beta_1 \right) = z + \beta_2 = \phi_2(z)$$

Thus $\psi\phi_1\psi^{-1} = \phi_2$, so ϕ_1 and ϕ_2 are conjugate. □

Lemma 0.4 (for Exercise III.6.3)**.** *Let f be a fractional linear transformation with a unique fixed point at ∞. Then f is a translation.*

Proof. Let $f(z) = \frac{az+b}{cz+d}$. Since f has a fixed point at $\infty, c = 0$. Since f has no finite fixed points, $d = a$. Thus $f(z) = \frac{a}{d}z + \frac{b}{d} = z + \frac{b}{d}$. □

Proposition 0.5 (Exercise III.6.3)**.** *A linear-fractional transformation with only one fixed point is conjugate to a translation.*

Proof. Let ϕ be a fractional linear transformation with a single fixed point w. Let ψ be a fractional linear transformation such that $\psi(w) = \infty$ (for example, $z \mapsto \frac{z}{z-w}$). Let y be a fixed point of $\psi\phi\psi^{-1}$. Then

$$\psi\phi\psi^{-1}(y) = y \implies \phi(\psi^{-1}(y)) = \psi^{-1}(y)$$

so $\psi^{-1}(y)$ is a fixed point of ϕ. Since ϕ has a unique fixed point, $\psi^{-1}(y) = w$, so $y = \psi(w) = \infty$. That is, $\psi\phi\psi^{-1}$ has a unique fixed point at ∞. Thus by the above lemma, $\psi\phi\psi^{-1}$ is a translation. Thus ϕ is conjugate to a translation. □

Proposition 0.6 (Exercise III.8.2)**.** *Let $z_1, z_2, z_3, z_4 \in \overline{\mathbb{C}}$ be distinct. They lie on a clircle if and only if the cross ratio $(z_1, z_2; z_3, z_4)$ is real.*

Proof. Let C be the unique clircle containing z_1, z_2, z_3, and let ϕ be the unique fractional linear trasformation such that

$$\phi(z_1) = \infty \qquad \phi(z_2) = 0 \qquad \phi(z_3) = 1$$

The unique clircle containing $0, 1, \infty$ is the line $\mathbb{R}\cup\{\infty\}$, so $\phi(C) = \mathbb{R}\cup\{\infty\}$ by preservation of clircles. Suppose that $z_4 \in C$. Since $\phi(z_1) = \infty$ and ϕ is injective, this implies $\phi(z_4) \in \mathbb{R}$. Now suppose that $(z_1, z_2; , z_3, z_4) = \phi(z_4)$ is real. Then $\phi^{-1}\phi(z_4) = z_4 \in C$. □

Lemma 0.7 (for Exercise III.9.2)**.** *Let $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a homeomorphism, and let $\gamma : I \to \mathbb{C}$ be a curve that separates $\mathbb{C}$ into two disjoint open, path-connected sets A, B; that is, $\mathbb{C} = A \cup B \cup \gamma(I)$, and $\phi \circ \gamma$ also splits $\mathbb{C}$ into two disjoint, open, path-connected sets C, D, that is, $\mathbb{C} = C \cup D \cup \phi \circ \gamma(I)$. Then $\phi(A) = C$ or $\phi(A) = D$.*

Proof. Suppose $\phi(A)\cap C \neq \emptyset$ and $\phi(A)\cap D \neq \emptyset$, so there exist $a_1, a_2 \in A$ with $\phi(a_1) \in C$ and $\phi(a_2) \in D$. Since A is path-connected, there is a path η connecting a_1 and a_2. Then $\eta \circ \phi$ is path connecting $\phi(a_1)$ and $\phi(a_2)$. However, a_1 and a_2 lie in distinct path components C, D so this is a contradiction. Thus $\phi(A) \subset C$ or $\phi(A) \subset D$. Relabelling if necessary, assume $\phi(A) \subset C$. By a similar argument applied to ϕ^{-1}, $\phi^{-1}(C) \subset A$ or $\phi^{-1}(C) \subset B$. Since ϕ is a bijection, we must have $\phi^{-1}(C) \subset A$. Thus $\phi(A) = C$. (Note that we may have relabelled, so it $\phi(A) = D$ is also possible.)

□

Lemma 0.8 (for Exercise III.9.2)**.** *The linear-fractional transformation that maps ∞ to 1 and has $i, -i$ as fixed points is given by*

$$z \mapsto \frac{z-1}{z+1}$$

Proof. We do a simple calculation to check:

$$\begin{aligned}
i &\mapsto \frac{i-1}{i+1} = \frac{i-1}{i+1}\left(\frac{1-i}{1-i}\right) = \frac{i-i^2-1+i}{1-i^2} = \frac{2i}{2} = i \\
-i &\mapsto \frac{-i-1}{-i+1} = \frac{-i-1}{-i+1}\left(\frac{i+1}{i+1}\right) = \frac{-i^2-i-i-1}{-i^2-i+i+1} = \frac{-2i}{2} = -i \\
\infty &\mapsto \frac{1}{1} = 1
\end{aligned}$$

□

Proposition 0.9 (Exercise III.9.2). *Let ϕ be the linear-fractional transformation that maps ∞ to 1 and has $i, -i$ as fixed points. The image of the disk $|z| < 1$ under ϕ is the right half-plane $\operatorname{Re} z > 0$, and the image of the half-plane $\operatorname{Re} z < 0$ under ϕ is the outside of the unit circle, $|z| > 1$.*

Proof. By the previous lemma,

$$\phi(z) = \frac{z-1}{z+1}$$

Here is a table of some values:

z	-4	-3	-2	-1	0	1	2	3	4
$\phi(z)$	$\frac{5}{3}$	2	3	∞	-1	0	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{5}$

Since ϕ takes $0, i, -i$ all to points on the unit circle, ϕ maps the whole imaginary axis to the unit circle. Since ϕ is a homeomorphism, by Lemma 0.7, the unit disk $|z| < 1$ must get mapped to one of the half planes $\operatorname{Re} z < 0$ or $\operatorname{Re} z > 0$. Since $\phi(-2) = 3$, ϕ maps the complement of the disk to the right half-plane, so the inside gets mapped to the left half-plane $\operatorname{Re} z < 0$.

The images of $-i, 0, i$ are $-i, -1, i$ respectively, so ϕ maps the imaginary axis to the unit circle. Since $\phi(-2) = 3$, the left half-plane $\operatorname{Re} z < 0$ gets mapped to the outside of the unit circle, $|z| > 1$. □

Lemma 0.10 (for Exercise III.9.3). *Let ϕ be a linear fractional transformation. Then there is a matrix inducing ϕ with determinant one.*

Proof. Let M be a matrix inducing ϕ. M is nonsingular, so $\det M \neq 0$. Then the matrix $\frac{1}{\det M} M$ also induces ϕ, and has determinant 1. □

The following is a purely topological, somewhat technical lemma. I just want to it so that I can say that a fractional linear transformation that restricts to a bijection on some subset of $\mathbb{C}$ must also restrict to a bijection on the boundary of that subset.

Lemma 0.11. *Let X be a topological space, and $A \subset X$. Let $\phi : X \to X$ be a homeomorphism so that $\phi(A) = A$ and $\phi|_A : A \to A$ is a homeomorphism. Then $\phi(\partial A) = \partial A$ and $\phi|_{\partial A} : \partial A \to \partial A$ is a homeomorphism.*

Proof. Let $x \in \partial A$. Then x is in the closure of A, so every open neighborhood U of x has non-empty intersection with A. Let V be an open neighborhood of $\phi(x)$. Then $\phi^{-1}(V)$ is an open neighborhood of x, so it has non-empty intersection with A. Since $\phi(A) = A$, this implies that V has non-empty intersection with A. Hence $\phi(x)$ is in the closure of A.

Now we show that $\phi(x)$ is not in the interior of A. Suppose $\phi(x)$ is in the interior of A. Then there is an open neighborhood V containing x that lies inside A. Then $\phi^{-1}(V)$ is an open neighborhood of x that lies inside A. This is a contradiction, since x is not in the interior of A. Hence $\phi(x) \in \partial A$. Thus $\phi(\partial A) \subset \partial A$.

Now we want to show that $\partial A \subset \phi(\partial A)$. Let $x \in \partial A$. Since ϕ is a homeomorphism from X to itself, there exists $y \in X$ so that $\phi(y) = x$. Let V be an open neighborhood of y. Then $\phi(V)$ is an open neighborhood of x, so $\phi(V)$ has non-empty intersection with A. Then since ϕ is a homeomorphism and a bijection $A \to A$, V has non-empty intersection with A. Thus y is in the closure of A.

Now we want to show that y is not in the interior of A. Suppose y is in the interior of A. Then there is an open set U containing y such that $U \subset A$. Then $\phi(U)$ is an open neighborhood of x contained in A, which contradicts that x is not in the interior of A. Hence y is not in the interior of A. Thus $y \in \partial A$, so $\partial A \subset \phi(\partial A)$.

Hence $\phi|_{\partial A} : \partial A \to \partial A$ is well-defined. It is a bijection because ϕ is a bijection, and it is continuous and has continuous inverse since ϕ and ϕ^{-1} are continuous. □

Lemma 0.12 (for Exercise III.9.3)**.** *Let ϕ be a fractional linear transformation that maps the upper half plane* $\operatorname{Im} z > 0$ *onto itself. Then ϕ maps* $\mathbb{R} \cup \{\infty\}$ *onto itself.*

Proof. Use the previous lemma with $X = \overline{\mathbb{C}}$ and $A = \{z : \operatorname{Im} z > 0\}$. □

Proposition 0.13 (Exercise III.9.3)**.** *A linear-fractional transformation maps the half-plane* $\operatorname{Im} z > 0$ *onto itself if and only if it is induced by a matrix with real entries whose determinant is 1.*

Proof. First consider a linear-fractional transformation induced by a matrix with real entries with determinant 1,

$$\phi(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. First, we can see that ϕ maps the extended real line onto itself. We compute

$$\phi(i) = \frac{ai + b}{ci + d} \left(\frac{d - ic}{d - ic} \right) = \frac{(bd + ac) + i(ad - bc)}{c^2 + d^2}$$

Since $ad - bc = 1$ and $c^2 + d^2 > 0$, this has positive imaginary part, so ϕ maps the upper half plane $\operatorname{Im} z > 0$ onto itself.

Now suppose that ϕ is a linear-fractional transformation that maps the upper half-plane $\operatorname{Im} z > 0$ onto itself. By the above lemma, ϕ maps $\mathbb{R} \cup \{\infty\}$ onto itself. Let $z_1, z_2, z_3 \in \mathbb{R}$ so that $\phi(z_i) \neq \infty$. Then let ω and η be the unique linear transformations such that

$$\begin{aligned} \omega(z_1) &= \infty & \omega(z_2) &= 0 & \omega(z_3) &= 1 \\ \eta(\phi(z_1)) &= \infty & \eta(\phi(z_2)) &= 0 & \eta(\phi(z_3)) &= 1 \end{aligned}$$

By uniqueness of threefold transitivity, $\omega = \eta \circ \phi$. Using the fact that z_1, z_2, z_3 are finite and their images under ϕ are also finite, Theorem III.5 of Sarason gives

$$\omega(z) = \frac{(z - z_2)(z_1 - z_3)}{(z - z_1)(z_2 - z_3)} \qquad \eta \circ \phi(z) = \frac{(\phi(z) - \phi(z_2))(\phi(z_1) - \phi(z_3))}{(\phi(z) - \phi(z_1))(\phi(z_2) - \phi(z_3))}$$

Setting these equal an solving for $\phi(z)$ gives

$$\phi(z) = \frac{az + b}{cz + d}$$

where

$$\begin{aligned}
a &= \phi(z_1)\phi(z_2)(z_2 - z_1) + \phi(z_1)\phi(z_3)(z_1 - z_3) + \phi(z_2)\phi(z_3)(z_3 - z_2) \\
b &= \phi(z_2)\phi(z_3)z_1(z_2 - z_3) + \phi(z_1)\phi(z_2)z_3(z_1 - z_2) + \phi(z_1)\phi(z_3)z_2(z_3 - z_1) \\
c &= \phi(z_1)(z_2 - z_3) + \phi(z_2)(z_3 - z_1) + \phi(z_3)(z_1 - z_2) \\
d &= \phi(z_1)z_1(z_3 - z_2) + \phi(z_2)z_2(z_1 - z_3) + \phi(z_3)z_3(z_2 - z_1)
\end{aligned}$$

Since each z_i and $\phi(z_i)$ are real, a, b, c, d are real, so ϕ is induced by a matrix with real entries. Any matrix inducing ϕ must be nonsingular, so we can form an equivalent matrix to induce ϕ by dividing each entry by the determinant $(ad - bc)$ to get a matrix of real entries and determinant one that induces ϕ. □

Proposition 0.14 (Exercise III.9.4)**.** *A linear-fractional transformation of the following form maps the disk $|z| < 1$ onto itself:*

$$\phi(z) = \frac{\lambda(z - z_0)}{\overline{z}_0 z - 1}$$

where $|z_0| < 1$ and $|\lambda| = 1$.

Proof. First suppose that ϕ has the above form. Let $|z| = 1$. Then

$$\begin{aligned}
|\phi(z)| &= \left|\frac{\lambda(z - z_0)}{\overline{z}_0 z - 1}\right| = |\lambda| \left|\frac{z - z_0}{\overline{z}_0 z - 1}\right| = \frac{|z - z_0|}{|\overline{z}_0 z - 1|} = \frac{|z - z_0|}{|\overline{z}_0 z - 1|}\left(\frac{|\overline{z}|}{|\overline{z}|}\right) \\
&= \frac{|z\overline{z} - \overline{z}z_0|}{|\overline{z}_0 z\overline{z} - \overline{z}|} = \frac{|1 - \overline{z}z_0|}{|\overline{z}_0 - \overline{z}|} = \frac{|\overline{z}z_0 - 1|}{|\overline{z} - \overline{z}_0|} = \frac{|z\overline{z}_0 - 1|}{|z - z_0|} = \frac{1}{|\phi(z)|}
\end{aligned}$$

Since $|\phi(z)|$ is a positive real equal to its own reciprocal, it is one. Hence $\phi(z)$ lies on the unit circle, so ϕ maps the unit circle to itself. Then notice that $\phi(0) = \lambda z_0$, which lies inside the unit circle. Then by Lemma 0.7, ϕ maps the disk $|z| < 1$ onto itself. □

Proposition 0.15 (Exercise IV.3.1)**.** *Let f be holomorphic in $\mathbb{C}$ and satisfies $f' = f$. Then f is a constant multiple of e^z.*

Proof. Define $g(z) = e^{-z}f(z)$. Then g is holomorphic, and by the product rule,

$$g'(z) = f(z)\frac{\partial}{\partial z}e^{-z} + e^{-z}\frac{\partial}{\partial z}f(z) = -f(z)e^{-z} + e^{-z}f(z) = 0$$

Then by Exercise II.8.1, g is constant in $\mathbb{C}$, say $g(z) = z_0$. Then

$$e^{-z}f(z) = z_0 \implies f(z) = z_0 e^z$$

□

(Exercise IV.5.2)
Define $f(z) = \exp(z^2)$. We want to describe the curves in $\mathbb{C} \cong \mathbb{R}^2$ defined by $|f| = c$ and $\arg f = c$ for a real constant c. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Then $z^2 = x^2 - y^2 + 2xyi$, and

$$\begin{aligned}
\exp(z^2) &= e^{x^2-y^2}\big(\cos(2xy) + i\sin(2xy)\big) \\
|\exp(z^2)| &= e^{x^2-y^2} \\
\arg\exp(z^2) &= 2xy
\end{aligned}$$

The curves $|\exp(z^2)| = c$ are curves of the form $x^2 - y^2 = c$. For $c = 0$, this is the union of the two lines $x = y$ and $x = -y$. For $c \neq 0$ this is a hyperbola, centered at $(0, 0)$, with a transverse axis being the x or y axis (depending on the sign of c) that is symmetric about both the x and y axes.

The curves $\arg\exp(z^2) = c$ are curves of the form $2xy = c$. For $c = 0$, this is the union of the two lines $x = 0$ and $y = 0$. For $c \neq 0$, this is a hyperbola, centered at $(0, 0)$, with transverse axis being either the line $x = y$ or $x = -y$ (depending on the sign of c).

(Exercise IV.5.3)
Define $f(z) = \exp\left(\frac{z+1}{z-1}\right)$. We want to describe the curves in $\mathbb{C} \cong \mathbb{R}^2$ defined by $|f| = c$ and $\arg f = c$ for a real constant c. Let $z = x + iy$ where $x, y \in \mathbb{R}$. Then

$$\begin{aligned}
\frac{z+1}{z-1} &= \frac{x+1+iy}{x-1+iy} = \frac{x+1+iy}{x-1+iy}\left(\frac{x-1-iy}{x-1-iy}\right) = \frac{x^2+y^2-1-2yi}{(x-1)^2+y^2} \\
&= \frac{x^2+y^2-1}{(x-1)^2+y^2} + i\frac{-2y}{(x-1)^2+y^2} \\
|f| &= \frac{x^2+y^2-1}{(x-1)^2+y^2} \\
\arg f &= \frac{-2y}{(x-1)^2+y^2}
\end{aligned}$$

The solutions to $|f| = c$ are points (x, y) satisfying

$$\frac{x^2+y^2-1}{(x-1)^2+y^2} = c \implies x^2 + y^2 - 1 = c(x-1)^2 + cy^2, \quad (x, y) \neq (1, 0)$$

For $c = 0$, we have

$$x^2 - 1 = x^2 - 2x + 1 \implies 0 = -2x + 1 \implies x = 1$$

so the solution curve is the line $x = 1$ with the point $(1, 0)$ omitted. For $c \neq 1$,

$$\begin{aligned}
& x^2 + y^2 - 1 = cx^2 - 2cx + 1 + cy^2 \\
\implies & (1-c)x^2 + 2cx + (1-c)y^2 = 2 \\
\implies & (1-c)\left(x + \frac{c}{1-c}\right)^2 + (1-c)y^2 = 2 + \frac{c^2}{c-1} \\
\implies & \left(x - \frac{c}{c-1}\right)^2 + y^2 = \frac{(c-1)^2+1}{(c-1)^2}
\end{aligned}$$

via completing the square. This is the equation of a circle with center $\left(\frac{c}{c-1}, 0\right)$ and radius $\sqrt{\frac{(c-1)^2+1}{(c-1)^2}}$. So the solution curve $|f| = c$ is (usually) a circle, with the point $(1, 0)$ omitted if necessary.

Now consider curves $\arg f = c$. These have the form

$$\frac{-2y}{(x-1)^2 + y^2} = c \implies -2y = c(x-1)^2 + cy^2, \quad (x, y) \neq (1, 0)$$

For $c = 0$, this is the line $y = 0$, with the point $(1, 0)$ omitted. For $c \neq 0$,

$$-2y = c(x-1)^2 + cy^2 \implies (x-1)^2 + \left(y + \frac{1}{c}\right)^2 = \frac{1}{c^2}$$

This is a circle with center $\left(1, \frac{-1}{c}\right)$ and radius $\frac{1}{c}$. So $\arg f = c$ is either the line $y = 0$ or a circle of this form, with the point $(1, 0)$ omitted if necessary.